TOPOLOGY COMPREHENSIVE EXAM: FALL 2014

There are SIX questions in this exam. All questions are compulsory. Time: 3 hours. All the best!

(1) (a) Define Reduced homology group $\widetilde{H}_n(X)$.
   (b) Prove that $\widetilde{H}_n(X) \simeq \widetilde{H}_{n+1}(SX)$ where $SX = X \times I/\sim$ with $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$.

(2) Clearly stating any theorems used, calculate all the cohomology groups of $F_2 \times S^2$ with coefficients in $\mathbb{Z}$, where F_2 is the closed oriented surface of genus 2.

(3) Can there exist a smooth map $\phi : S^5 \to \mathbb{R}^5$ such that the induced map at tangent spaces is injective at all points of S^5?

(4) (a) Consider the 2-form $\omega := xdz \wedge dy + x^2 dx \wedge dy$ on $\mathbb{R}^3$. Is this a smooth 2-form? (State the definition of a smooth 2-form and check whether or not this is smooth)
(b) Consider the natural inclusion $S^2 \subset \mathbb{R}^3$. Apply Stokes theorem and compute $\int_{S^2} \omega|_{S^2}$. (Just state without proof the choice of your manifold with boundary) Work with the orientation on $\mathbb{R}^3$ given by $dx \wedge dy \wedge dz$.

(5) Let G be a compact connected Lie group. It is a fact that G is a finite simplicial complex and so one may apply Lefschetz fixed point theorem. Use the Lefschetz fixed point theorem to show that the Euler characteristic of G is 0.

(6) (a) Let $X = T^2 \setminus B^2$ be the space obtained by removing an open disk from a torus. Let $x_0 \in \partial X$. Calculate the fundamental group $\pi_1(X, x_0)$.
   (b) Let Mob be a mobius strip. Let $x_1 \in \partial Mob$. Calculate the fundamental group $\pi_1(Mob, x_1)$.
   (c) Let $\gamma : \partial Mob \to \partial X$ be the identity map of S^1 taking x_1 to x_0. Let $Y = X \sqcup Mob/\sim$ where $x \sim \gamma(x)$ for all $x \in \partial Mob$, i.e., Y is obtained by sticking the boundary of a mobius strip to the boundary of a punctured torus. Calculate the fundamental group $\pi_1(Y, [x_0])$.